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DERIVATIONS OF PI-ALGEBRAS AND SMOOTHNESS OF NONCOMMUTATIVE CURVES

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0. Introduction

If C is an affine algebra over a fixed field k, which is assumed to be algebraically closed, then the affine k-variety $X = \mathrm{Spec}(C)$ associated to C is smooth if for each $p \in X$ the module of differentials $\Omega_{C_p/k}$ is free over C_p, the local ring at p. In particular, it follows that $\mathrm{Der}_k(C_p, C_p)$ is a free C_p-module.

In [7] Lipman studies the converse problem: if $\mathrm{Der}_k(C_p, C_p)$ is free, what can one say about C_p, or more specifically: is C_p regular? It is easy to see that the last question must be answered in the negative if k has nonzero characteristic. In characteristic zero, however, the problem remains open. In particular, it has been established in [7] that in case $\mathrm{char}(k)=0$, if $\mathrm{Der}_k(C_p, C_p)$ is free, then C_p is integrally closed, hence for curves that C_p is regular.

In the noncommutative case the situation is completely different, due to the absence of a well-behaving module of differentials. In this note we thus adopt a somewhat different point of view. We define a prime pi-algebra R to be preregular if the space $\mathrm{Der}_k(R, R)$ is free over the center C of R and of the exact rank, see below. This notion seems to lead to the expected properties and behaves particularly well, when applied to curves. In higher dimensions however, it is clear that a stronger notion of regularity may be necessary in order to obtain a meaningful generalization of the commutative case, we hope to return to this problem in a subsequent paper. After some preliminaries concerning derivations, in particular the extension properties of derivations from a commutative ring to an Azumaya algebra over it, we study derivations of pi-algebra. A useful result states that if F is a finitely separably generated field extension of k of transcendence degree t and Q a central simple F-algebra of degree n, then $\dim_F \mathrm{Der}(Q, Q) = n^2 + t - 1$. This is easily seen to extend to Azumaya algebras, which are free modules over their center.

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We define a noncommutative affine k-variety $X=\Omega(R)$ to be preregular if $Q_p^{\mathrm{bi}}(R)$, cf. [15], is a pre-regular k-algebra, for all $p\in X$, i.e. its space of k-derivations is free of rank $\dim_F \mathrm{Der}_k(Q,Q)$ over $Z(Q_p^{\mathrm{bi}}(R))$, where Q is the function ring of X. It is thus clear that any Azumaya algebra over a preregular commutative local k-algebra is preregular in this sense, so points of $\Omega(R)$ of maximal pi-degree which lie over a preregular central point are certainly preregular. Actually, we show in the last section that for noncommutative curves preregularity may be studied by looking at the center. In general the preregular points of an algebraic k-variety for a dense set, but it appears there may exist preregular points of the noncommutative variety which lie over a non-preregular central point, these points then necessarily having non-maximal pi-degree.

1. Generalities

1.1. Throughout R is a prime pi-algebra, which we will assume to be affine over an algebraically closed field k. Its center will be denoted by C. For any two-sided R-module M we let $\mathrm{Der}_k(R,M)$ denote the k-vector space of all k-derivations $D:R\to M$, i.e. the k-linear maps $D:R\to M$ such that $D|k=0$ and $D(rs)=D(r)s+rD(s)$for all $r,s\in R$. Clearly $\mathrm{Der}_k(R,M)$ is a C-module.

Recall that a ring morphism $\varphi:R\to S$ is said to be an *extension* in the sense of Procesi [9], if S is generated by $\varphi(R)$ and $Z_R(S)$, the R-centralizer of S, which consists of all $s\in S$ such that $\varphi(r)s=s\varphi(r)$ for $r\in R$. If S is generated by its center $Z(S)$ over $\varphi(R)$, then we speak of a *central extension*. An extension $\varphi:R\to S$ maps $Z(R)$ to $Z(S)$.

1.2. Lemma. *Let $\varphi:R\to S$ be a ring extension and let $D\in\mathrm{Der}_k(R,S)$. Then*

(1.2.1.) *D maps $Z(R)$ into $Z_R(S)$.*

(1.2.2.) *If φ is a central extension, then D maps $Z(R)$ into $Z(S)$.*

Proof. Pick $c\in Z(R)$ and $r\in R$, then $D(cr)=D(rc)$, hence $\varphi(c)D(r)+D(c)\varphi(r)=\varphi(r)D(c)+D(r)\varphi(c)$. Hence $D(c)\varphi(r)=\varphi(r)D(c)$, as $\varphi(c)\in Z(S)$, therefore $D(c)\in Z_R(S)$, which proves (1.2.1.). The assertion (1.2.2.) is an easy consequence of (1.2.1.). □

For a detailed exposition of the properties of localization at symmetric kernel functors, which we will use freely in this paper, we refer to [13, 14, 15]. Let us just point out the following easy fact:

1.3. Lemma. *Let $D:R\to R$ be a derivation and let φ be a symmetric kernel functor in R-mod with idempotent filter $L(\sigma)$. For each $I\in L(\sigma)$ there exists an $H\in L(\sigma)$ such that $H\subset I$ and $D(H)\subset I$.*

Proof. Since σ is symmetric we may assume I to be a (two-sided!) ideal of R. Put $H=I^2$, then if $i,j\in I$ we have $D(ij)=D(i)j+iD(j)\in I$, hence H does the trick. □

1.4. Remark. Note that for any ideal I of R we have that $I+D(I)$ is an ideal of R. Indeed, for all $r\in R$ and $i\in I$ we have $rD(i)\in I+D(I)$ and $D(i)r\in I+D(I)$, because $rD(i)=-D(ri)+D(ri)$ and similarly for $D(i)r$.

Let us recall some generalities about derivations of commutative fields.

1.5. Proposition. *Let F be a finitely separable generated field extension. Then $\dim_F \Omega_{F/k}=\operatorname{tr}\deg_k F$, where Ω denotes the F-space of k-differentials.*

It follows that for any finite dimensional F-algebra K we have that $\dim_F \operatorname{Der}_k(F,K)=\operatorname{tr}\deg_k F\cdot\dim_F K$ and $\operatorname{rk}_K \operatorname{Der}_k(F,K)=\operatorname{tr}\deg_k F$. □

In particular, if X is an algebraic k-variety with function F over a perfect field k, then $\dim X=\dim_F \operatorname{Der}_k(F,F)$.

It is sometimes possible to extend derivations of the center of R to derivations of R, let us just mention

1.6. Proposition. *Let A be a ring which is projective over its center and of Hochschild dimension one. Let M be a two-sided A-module, then every derivation $d:Z(A)\to Z_A(M)$ where $Z_A(M)=\{m\in M \mid am=ma \text{ for all } a\in A\}$ extends to a derivation $D:A\to M$.*

Proof. Cf. [3]. □

1.7. Corollary. *If A is an Azumaya algebra over $Z(A)=C$, then any derivation $d:C\to C$ extends to a derivation $D:A\to A$ and for any A^e-module M there is a canonical isomorphism $H^1(A,M)=\operatorname{Der}(Z(A),Z_A(M))$.* □

1.8. Obviously in Corollary 1.7 the derivation $D:A\to A$ of $d:C\to C$ is far from being unique in general, since e.g. inner derivations of A induce the trivial derivation on $Z(A)$. Note also that any C-derivation $A\to M$ is necessarily inner.

2. Derivations of pi-algebras

2.1. Let R be a prime pi-algebra. Every derivation $D:R\to R$ extends uniquely to a derivation $D':Q(R)\to Q(R)$. Indeed, by Posner's Theorem $Q(R)$ is obtained from R by inverting central elements and it is then completely clear how D' is defined: if $r\in R$ and $c\in C$, then $D(rc^{-1})=(-D(r)c+rD(c))c^{-2}$ as expected. We may actually sharpen this observation as follows:

2.2. Lemma. *Let R be a prime pi-algebra and σ a symmetric kernel functor in R-mod. There is a unique derivation $D_\sigma : Q_\sigma(R) \to Q_\sigma(R)$ extending D.*

Proof. Let D' be the extension of D to $Q(R)$ as described above and put $D_\sigma = D' \mid Q_\sigma(R)$. Pick $q \in Q_\sigma(R)$ and let $I \in L(\varphi)$ be such that $Iq \subset R$. According to Lemma 1.3 we may take $H \in L(\varphi)$ with $H \subset I$ such that $D(H) \subset I$. For $h \in H$ we obtain $D(hq) = hD'(q) + D(h)q$. Since $hq \in R$, we have that $D(hq) \in R$. Now $D(h) \in I$ yields that $hD'(q) = D(hq) - D(h)q \in R$. Thus $HD'(q) \subset R$, but since $H \in L(\sigma)$, it follows that $D'(q) \in Q_\sigma(R)$. Moreover, it is clear that D_σ unique as such. □

2.3. Lemma. *Let K be a commutative ring and let Λ be an Azumaya algebra over K. Let A be an algebra over Λ which is free of rank p as an $F = Z_A(\Lambda)$-module, then* $\mathrm{Der}_K(\Lambda, A)$ *is a free F-module of rank $p-1$.*

Proof. Since Λ is separable over K every K-derivation $\Lambda \to A$ is inner. Choose a basis $\{e_1 = 1, e_2, \ldots, e_p\}$ for A over F and let d_i be the inner derivation $[e_i, -]$.

We claim that $\{d_2, \ldots, d_p\}$ is an F-basis for $\mathrm{Der}_k(\Lambda, A)$. Indeed, if these d_i were F-independent, then we would find $\lambda_i \in F$, not all vanishing, such that $\sum_{i=2}^{p} \lambda_i d_i = 0$, but then $\sum_{i=2}^{p} \lambda_i e_i$ commutes with everything in Λ, i.e. we may find $\lambda \neq 0$ in $F - \{0\}$ such that $\sum \lambda_i e_i + \lambda e_1 = 0$, a contradiction. On the other hand, it is clear that $\{d_2, \ldots, d_p\}$ generates $\mathrm{Der}_k(\Lambda, A)$ as an F-module, proving the assertion. □

2.4. Corollary. *Let C be a commutative ring and n a positive integer. Then* $\mathrm{Der}_C(M_n(C), M_n(C))$ *is free of rank $n^2 - 1$ over C.* □

2.5. Proposition. *Let Q be a central simple F-algebra, where F is a finitely separably generated field extension of k. Then* $\dim_F \mathrm{Der}_k(Q, Q) = \dim_F Q + \mathrm{tr\,deg}_k F - 1$.

Proof. Choose a separating transcendence basis for F/k, say $\{x_1, \ldots, x_t\}$ and put $K = k(x_1, \ldots, x_t)$. Since F is separably algebraic over K, we have that $\Omega_{F/k} = \Omega_{K/k} \otimes_K F$ and hence

$$\mathrm{Der}_k(F, F) = \mathrm{Hom}_F(\Omega_{F/k}, F) = \mathrm{Hom}_F(\Omega_{K/k} \otimes_K F, F)$$

$$= \mathrm{Hom}_K(\Omega_{K/k}, K) \otimes_K F = \mathrm{Der}_k(K, K) \otimes_K F,$$

hence $\dim_F \mathrm{Der}_k(F, F) = \dim_K \mathrm{Der}_k(K, K) = t$. Let $\{D_1, \ldots, D_t\}$ be an F-basis for $\mathrm{Der}_k(F, F)$. Since Q is central simple over F, each of these D_i $(i = 1, \ldots, t)$ extends to a k-derivation on Q, say $D_i' : Q \to Q$. Now let $D : Q \to Q$ be an arbitrary derivation then by Lemma 1.2 we know that D maps F into itself, hence we have $D = \sum_{i=1}^{t} \lambda_i D_i$ for some $\lambda_i \in F$. It follows that $D - \sum \lambda_i D_i'$ is a k-derivation of Q which vanishes on F, i.e. an F-derivation. But all F derivations of Q are inner hence we may find $q \in Q$ such that $D = \sum \lambda_i D_i' + [q, -]$. Let $\{e_1 = 1, e_2, \ldots, e_{n^2}\}$ be a basis for Q over F, where $n^2 = \dim_F Q$, then q may be written as $\sum \mu_j e_j$ for some μ_j in F. It follows that

$$D = \sum_{i=1}^{t} \lambda_i D_i' + \sum_{j=2}^{n^2} \mu_j [e_j, -],$$

and $\{D_1', \dots, D_t', \dots, [e_2, -], \dots, [e_{n^2}, -]\}$ generates $\mathrm{Der}_k(Q, Q)$ over F. Conversely, assume that we may find $\lambda_i, \mu_j \in F$ as above such that

$$0 = \sum_{i=1}^{t} \lambda_i D_i' + \sum_{j=2}^{n^2} \mu_j [e_j, -].$$

Then for each $f \in F$ we have that $0 = \sum \lambda_i D_i'(f) = \sum \lambda_i D_i(f)$, i.e. $\sum \lambda_i D_i$ vanishes identically on F, so $\lambda_1 = \dots = \lambda_t = 0$. Now, from $\sum_{j=2}^{n^2} \mu_j [e_j, -] = 0$, it follows by Lemma 2.3 that $\mu_2 = \dots = \mu_{n^2} = 0$, hence the set $\{D_1', \dots, D_k', [e_2, -], \dots, [e_{n^2}, -]\}$ is actually a basis for $\mathrm{Der}_k(Q, Q)$ over F. This proves the assertion. □

2.6. Corollary. *Let R be an Azumaya algebra over C and assume that* $\mathrm{Der}_k(C, C)$ *is a free C-module; if R is free over C, then* $\mathrm{Der}_k(R, R)$ *is free as a C-module and* $\mathrm{rk}_C \mathrm{Der}_k(R, R) = \mathrm{rk}_C R + \mathrm{rk}_C \mathrm{Der}_k(C, C)$.

Proof. This follows along the lines of the proof of Proposition 2.5. □

Note. We will derive below a generalization of this result to arbitrary pi-algebras.

2.7. At this point, let us motivate the definitions we are about to introduce by recalling some commutative results. Let X be a (commutative) algebraic k-variety with sheaf of differentials $\Omega_{X/k}$, then X is nonsingular iff $\Omega_{X/k}$ is locally free of rank $n = \dim X$. It follows in particular that $\mathbf{Hom}_{\mathcal{O}_X}(\Omega_{X/k}, \mathcal{O}_X)$ is locally free of rank n. If X is affine with coordinate ring R, then:

$$\mathbf{Hom}_{\mathcal{O}_X}(\Omega_{X/k}, \mathcal{O}_X) = \mathrm{Hom}_R(\Omega_{R/k}, R)^{\sim} = \mathrm{Der}_k(R, R)^{\sim},$$

i.e. locally the dual of $\Omega_{K/k}$ is just the space of k-derivations of the ring of sections into itself. If S is a multiplicative subset of R, then $S^{-1}\mathrm{Der}_k(R, R) = \mathrm{Der}_k(S^{-1}R, S^{-1}R)$, so indeed $(\mathrm{Der}_R(R, R))^{\sim}(X_f) = \mathrm{Der}_k(\tilde{R}(X_f), \tilde{R}(X_f))$ for $f \in R$ and $\mathcal{M}_X = \Omega_{X/k}$, the $\mathcal{O}_X$ = dual of $\Omega_{X/k}$ is affine. Let us work locally for a moment, i.e. R is a local ring of a point p on an algebraic k-variety X; let $D = \mathrm{Der}_k(R, R)$. Then it is tempting to ask whether D is free over R implies P to be a regular point. In characteristic $p \neq 0$ it is easy to see that this is not true, whereas in characteristic 0 the question seems to be open, cf. [7]. However, in loc. cit. Lipman has proved that points P at which D is free are necessarily normal in the latter case, i.e. smooth if X is a curve.

2.8. Let us now consider an affine prime pi-algebra R over k, and let $\Omega(R)$ be the affine k-variety corresponding to R, cf. [15]. It is endowed with a sheaf of rings defined as follows. Let $X(I)$ be a Zariski open subset of $\Omega(R)$ and let $\mathscr{L}(I)=\{L \lhd R \mid I \subset \operatorname{rad} L\}$ be the associated filter of (twosided) ideals, then one defines $Q_I(R)$ to consist of all $q \in Q(R)$ for which an $L \in \mathscr{L}(I)$ may be found with the property that $Lq \subset R$. Finally, put $Q_I^{\mathrm{bi}}(R)=R \cdot Z_R(Q_I(R))$. These $Q_I^{\mathrm{bi}}(R)$ may be used to define a sheaf $\mathcal{O}_R$ on $\Omega(R)$ with the property that for each $P \in \Omega(R)$ we have $\mathcal{O}_{R,P}=Q_{R-P}^{\mathrm{bi}}(R)$. Here, as above, $Q_{R-P}^{\mathrm{bi}}(R)=R \cdot Z_R(Q_{R-P}(R))$ and $Q_{R-P}(R)$ consists of all $q \in Q(R)$ such that there is an ideal $I \not\subset P$ with $Iq \subset R$. Since $Q_{R-P}^{\mathrm{bi}}(R)$ contains information on the local behaviour of $\Omega(R)$ at P, we are led to

2.9. Definition. Let R be a prime pi-algebra over k which has the property that $\operatorname{tr} \deg_k Q(Z(R))=t<\infty$ (e.g. a localization of an affine prime pi-algebra). Then we call R a *preregular* k-algebra if $\operatorname{Der}_k(R,R)$ is free over $Z(R)$ of rank n^2-1+t, i.e. in particular

$$\operatorname{rk}_{Z(R)} \operatorname{Der}_k(R,R)=\dim_k \operatorname{Der}_k(Q(R),Q(R)).$$

If $X=\Omega(R)$, where R is an affine prime pi algebra over k, then we say that X or R is *preregular at* $P \in X$ if $Q_P^{\mathrm{bi}}(R)$ is a preregular k-algebra. We call X *preregular* if it is preregular at all $P \in X$.

2.10. Lemma. *Let C be a commutative preregular local k-algebra. Then any Azumaya algebra R over C is a preregular (quasi-local) k-algebra.*

Proof. Let C be preregular, then we may find a C-basis $\{\delta_1,\dots,\delta_t\}$ for $\operatorname{Der}_k(C,C)$, where $t=\operatorname{tr} \deg_k Q(C)$. As R is an Azumaya algebra over C, it follows from Corollary 1.7) that each of these δ_i extends to a derivation $\Delta_i: R \to R$. On the other hand, since C is local, R is free over C. The result now follows from Corollary 2.6. □

2.11. Corollary. *For X as above, the set of preregular points of X contains an open subset of X and hence is certainly dense in X.*

Proof. Let FR denote the Formanek center of R, then it is well known that for any $P \in X(FR)$ we have $Q_{R-P}^{\mathrm{bi}}(R)=R_p$, where $p=P \cap Z(R)$ and that $Q_{R-P}^{\mathrm{bi}}(R)$ is an Azumaya algebra of degree n over its local center C_p. The result then follows immediately from the commutative counterpart, restricting to primes not containing FR. □

3. Curves

3.1. Let R be an affine prime pi-algebra, then we call R a curve if it has Krull dimension one. We will sometimes refer to $\Omega(R)$ as a curve in this situation. It is well known that if R is a curve, then R is a finite module over its center C and C is an affine domain of Krull dimension one too. Moreover, R is an order if and only if C is a Dedekind domain. Now if we denote by K the quotient field of C, then $Q(R)=M_n(K)$ where $Q(R)$ is the classical simple ring of fractions of R and n its pi-degree. Indeed, since R is a curve, by definition $\operatorname{tr}\deg_k K=\dim R=1$, so we may apply Tsen's Theorem. For more details cf. [2, 15].

3.2. Lemma. *Let R be a prime pi-ring whose center C is a Dedekind domain, then for all $P\in \operatorname{Spec}(R)$ and $p=P\cap C$ we have:*

(3.2.1.) $Z(Q_{R-P}(R))=C_p$,

(3.2.2.) $Q^{\text{bi}}_{R-P}(R)=R_p$.

Proof. The second statement follows trivially from the first. Now, clearly $C_p\subset Z(Q_{R-P}(R))\subset Q(C)$ and C_p is a discrete valuation ring, hence $C_p=Z(Q_{R-P}(R))$ as C_p is a maximal subring of $Q(C)$. □

3.3. Proposition. *Let R be an affine prime pi-algebra and let $P\in\Omega(R)$, then the following statements are equivalent:*

(3.3.1.) *R is a preregular at P.*

(3.3.2.) $\operatorname{Der}_k(Q^{\text{bi}}_{R-P}(R), Q^{\text{bi}}_{R-P}(R))$ *is a free* $Z(Q^{\text{bi}}_{R-P}(R))$*-module.*

Proof. Let us write S for $Q^{\text{bi}}_{R-P}(R)$ and D for $Z(Q^{\text{bi}}_{R-P}(R))$. By definition (3.3.1.) implies (3.3.2.). Conversely, let $Q=Q(R)=Q(S)$ and $K=Q(C)=Z(Q)$. Then Q is a central simple K-simple algebra of dimension n^2. Pick a basis $\{e_1,\dots,e_{n^2-1},e_{n^2}=1\}$ of Q contained in R and for $1\le\alpha n^2-1$, let $d_\alpha=[e_\alpha,-]$, the inner derivation of Q determined by e_α. One easily sees that the d_α form a C-linearly independent subset of $\operatorname{Der}_k(R,R)$. On the other hand, since K is finitely separately generated over k, we have $\dim_K \operatorname{Der}_k(Q,Q)=n^2-1+t$, where $t=\operatorname{tr}\deg_k K$, cf. Proposition 2.5. So there are K-linearly independent $D_1,\dots,D_t\in\operatorname{Der}_K(Q,Q)$, which are nontrivial on K. Pick a finite set of k-algebra generators $\{f_1,\dots,f_n\}$ for R and choose $0\ne c\in C$ such that $cD_if_j\in R$ for $1\le i\le t$, $1\le j\le p$. Let D_i' be the unique extension of cD_i to $Q_{R-P}(R)$, cf. Lemma 2.2. Then obviously D_i' maps $Q^{\text{bi}}_{R-P}(R)$ into itself as $D_i'(Z_{R-P}(R))\subset Z_R(Q_{R-P}(R))$, and it is clear that $\{D_1',\dots,D_t'\}$ forms a C-linearly independent subset of $\operatorname{Der}_k(S,S)$. Since by assumption $\operatorname{Der}_k(S,S)$ is a free D-module, we thus find that $\operatorname{rk}_D\operatorname{Der}_k(S,S)\ge n^2-1+t$. Now, by 2.1, there is a canonical D-linear injective map $\operatorname{Der}_k(S,S)\to\operatorname{Der}_k(Q,Q)$, which is localizing yields that $\operatorname{rk}_D\operatorname{Der}_k(S,S)\le n^2-1+t$ as well. This proves the assertion. □

3.4. Corollary. *Let R be a curve with center C; if C is regular, then R is preregular.*

Proof. Since C is a Dedekind domain, we know that R is a finite C-module, being an order. On the other hand, from Lemma 3.2, it follows that $Q^{\text{bi}}_{R-P}(R)=R_p$, by central localization at $P\cap C=p$. Hence $Q^{\text{bi}}_{R-P}(R)=S$ is a finite module over $D=C_p=Z(S)$. Since R is prime, S is certainly torsionfree over D; hence S is a free

D-module of finite rank. But then, since D is noetherian, it follows that $\mathrm{Der}_k(S,S)$ is a finite D-module, which is free as it is torsion-free. It now suffices to apply Proposition 3.3 to finish the proof. □

3.5. Corollary. *Let R be a quasi-local prime pi-algebra with noetherian center C and finite (left) global dimension. If* $\mathrm{tr\,deg}_k\, Z(Q(R)) = 1$, *then R is preregular.*

Proof. Since C is noetherian, it follows from [5] that R is left noetherian and finite as a C-module, hence R is a (left) order in $Q(R)$, and $Q(R)=M_n(F)$ where $F=Q(C)$ and n is the pi-degree of R, by Tsen's Theorem. Since the left global dimension is finite and R is left noetherian, the fact that R is a quasi-local order in $M_n(F)$ implies R to be of the form $R=M_n(C)$. But, since R is a central C-algebra which is finitely generated and torsion free over C, it follows that C is a Dedekind domain, due to a result of Harada's. Now, as in Corollary 3.4, it follows that $\mathrm{Der}_k(M_n(C), M_n(C))$ is certainly free as a C-module, proving the assertion, in view of Proposition 3.3. □

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